On Self-Excited Oscillations in Circuits with Iron-Core Coils.*

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Abstract

The spontaneous generation of oscillations in circuits with iron-core coils is treated with reference to an analogous mechanical problem, in that a periodic variation of the magnetizing current in iron-cored coils leads to a proportionately varying inductance.

It has been known for quite a while that, under certain conditions, oscillating circuits containing a closed iron-core coil, which is driven by an exterior sinusoidal EMF, can self-excite additional oscillations whose frequency is not necessarily equal to that of the exterior EMF nor a harmonic of it. K. Heegner[1] has experimentally investigated the establishment of such well-known oscillations and, with the aid of energy considerations, has also obtained theoretically determined expressions (especially concerning the existing frequency relations).

In this publication it will be demonstrated that the most important of these phenomena (that is, self-excitation) also permits a variant of Heegner's representation. I was led to this through Lord Rayleigh's[2] work on mechanics in which an example of frequency splitting is treated. Because of its simplicity, it seems to me that the application of this method to our problem is not without significance.

The gist of Rayleigh's work should be briefly given here since it facilitates an understanding of the electromagnetic analogy:

A taught chord is attached to the end of a (tuning) fork's prong, which oscillates at frequency ω_o in the direction of the chord. Consequently, the tension of the chord experiences a periodic variation at the same frequency. Initially, there is no cause present for transversal vibrations. However, under certain conditions, the chord actually executes powerful transverse oscillations[3] at a frequency of $\omega_o/2$. For the explanation of the system's transverse oscillation phenomenon, Rayleigh formed the equation[i]

$$\frac{d^2 u}{d t^2} + \alpha \frac{du}{dt} + n^2 \bullet u - 2\beta \sin \omega_o t \bullet u = 0$$

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* Translated by J.F. Corum, National Electrodynamics, July 31, 2007.

1 K. Heegner, "Self-Excited Phenomena in Systems with Distorted Superposition." Zeitschr. für Phys., Vol. 29, p. 91, 1924; and Vol. 33, p. 85, 1925. Ref. Jahrbuch für drahtlosen Telegraphie und Telephonie (Zeitschrift für Hochfrequenztechnik), Vol. 27, p. 30,1926.

2 "On Maintained Oscillations," Phil. Mag., Vol. 15, p. 229, 1883; also see Rayleigh, Theory of Sound, Vol. 1, p. 82, London, 1926.

3 The previously cited reference gives a series of further examples from mechanics. In this connection, also see: B. van der Pol, "Stabiliseering door kleine Trillingen," Physica, Nederl. Tijdschrift v. Natuurkde, 5, p. 157, 1925.

and pointed out that, by a specific choice of the constants α, β, n^2, this differential equation, possesses a particular integral

$$u = A \bullet \sin\left(\frac{\omega_o}{2} t - \phi\right) .$$

The electromagnetic analogy to this mechanical problem is usually formed as an oscillating circuit whose capacity or inductance varies periodically in time. The latter is the case if the circuit contains an iron-cored coil that is magnetized by an alternating current. By this means the inductance of the coil undergoes a periodic variation (see below) of the *same* frequency as the magnetizing alternating current for the case that direct current is additionally superimposed, but of *double frequency* if the direct current component is absent.

It will now be shown that self-excitation in these kinds of systems, which are analogous to mechanical examples, can be explained from the periodicity of the self-induction.

The following section will next present the self-induction of an iron-cored coil as a periodic function of time. In the subsequent sections, self-excitations in simple oscillating circuits with iron-cored coils (with and without DC bias-magnetization) will be treated, and, finally, an extension of Rayleigh's formulation leads to a description of *coupled* systems with iron-cored coils.

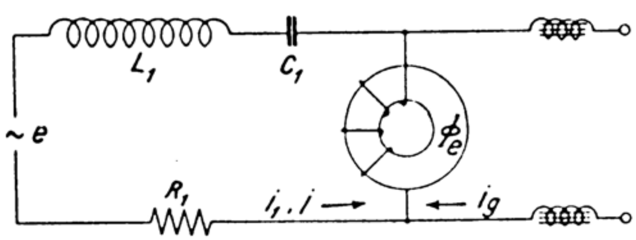

Figure 1. Oscillation circuit with an iron-cored coil and DC bias-magnetization.

1. An Iron-cored Coil as a Periodically Varying Inductance.

Consider a coil with a closed iron-core, with magnetic flux Φ_e that has magnetic bias by DC current i_g, in an oscillation circuit with capacitance C_1. Let the inductance be L_1 and the dissipation resistance be R_1. (Figure 1) Let an exterior sinusoidal electromotive force e, of frequency ω_o be applied to the circuit. Therefore, the differential equation for the circuit is

$$\left. \begin{aligned} &\frac{d\Phi_e}{d i_1} \bullet \frac{d i_1}{d t} + L_1 \bullet \frac{d i_1}{d t} + i_1 R_1 + \frac{1}{C_1} \int i_1 \, dt \\ &\quad = E \sin(\omega_o t - \phi) = e \end{aligned} \right\} \quad (1)$$

If the iron-free inductance of L_1 is sufficiently great, the higher harmonics of i_1 can be neglected in comparison to the fundamental oscillation, and Equation (1) possesses a stationary oscillation as a solution:

$$i_1 = J_1 \sin(\omega_o t)$$

as is commonly attained if an inductance L_e for the iron-cored coil, defined according to Schunck-Zenneck,[4] is introduced.

According to the method of small vibrations, an additional current i should now be superposed on the current i_1 with the requirement that $i << i_1$. Because of this assumption, in the first approximation the differential equation for the varying current is:

$$\begin{aligned} &\frac{d\,\Phi_e(i_1+i)}{d\,(i_1+i)} \bullet \frac{d\,(i_1+i)}{d\,t} \\ &= \frac{d\,\Phi_e(i_1)}{d\,(i_1)} \bullet \frac{d\,(i_1+i)}{d\,t} + i \bullet \frac{d}{d\,t}\left(\frac{d\,\Phi_e(i_1)}{d\,i_1}\right), \end{aligned} \tag{2}$$

$$\left.\begin{aligned} &\frac{d\,\Phi_e(i_1)}{d\,i_1} \bullet \frac{d\,(i_1+i)}{d\,t} + i \bullet \frac{d}{d\,t}\left(\frac{d\,\Phi_e(i_1)}{d\,i_1}\right) + L_1 \bullet \frac{d\,(i_1+i)}{d\,t} \\ &\quad + R_1\,(i_1+i) + \frac{1}{C_1}\int (i_1+i)\,dt = e \end{aligned}\right\}$$

Therefore, the equation for the superposed oscillations is:

$$\frac{d\,\Phi_e}{d\,i_1} \bullet \frac{d\,i}{d\,t} + i \bullet L_1 \frac{d\,i}{d\,t} + i \bullet \frac{d}{d\,t}\left(\frac{d\,\Phi_e}{d\,i_1}\right) + i \bullet R_1 + \frac{1}{C_1}\int i\,dt = 0\,. \tag{3}$$

The coefficients $\frac{d\,\Phi_e}{d\,i_1}$ and $\frac{d}{d\,t}\left(\frac{d\,\Phi_e}{d\,i_1}\right)$ are now functions of i_1, and are, therefore, periodic functions of time,[5] and can, as shown below, be represented by the following Fourier Series:

$$\frac{d\Phi_e}{d\,i_1} = L_o + k_1 \sin\omega_o t + k_2 \sin 2\,\omega_o t + k_3 \sin 3\,\omega_o t + \ldots \tag{4}$$

and, therefore,

$$\begin{aligned} \frac{d}{d\,t}\left(\frac{d\,\Phi_e}{d\,i_1}\right) &= \omega_o\,k_1 \cos\omega_o t - 2\,\omega_o\,k_2 \sin 2\,\omega_o t \\ &\quad + 3\,\omega_o\,k_3 \cos 3\,\omega_o t - \,.\,.\,. \end{aligned} \tag{5}$$

In these equations, L_o and k_n are functions of the amplitude J_1 of the magnetizing alternating current and of the superposed direct current, i_g.

According to Rayleigh, under certain conditions, owing to the periodicity of the coefficients, Equation (3) will possesses stationary solutions for i, which produce the original superposed solution of Equation (1) for i_1 and the consequent self-excitation of our system. For this investigation, the coefficients of the series in Equation (5) are to be calculated as functions of J_1 and i_g:

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[4] This Jahrbuch, Vol. 19, p. 117, 1922. [See Endnote Reference ii.]

[5] In Equation (3), $d\Phi_e/di_1$ has the meaning of a periodically varying self-induction.

a) L_o and k_n in the case of pure alternating current magnetization ($i_g = 0$).

As usual, one may employ the formulation of Dreyfuss[ii] for the analytical representation of the magnetization curve of iron:

$$\Phi_e = L_m \, arctg\,(q \bullet i_1) + L' i_1 \tag{6}$$

where, as above, $i_1 = J_1 \sin \omega_o t$. So, first of all we have:

$$\frac{d\Phi_e}{d\,i_1} = \frac{L_m}{1 + q^2\, i_1^2} + L' = \frac{L_m}{1 + q^2\, J_1^2\, \sin^2 \omega_o t} + L' \tag{7}$$

From this it follows that $d\Phi_e/di_1$ possesses a fundamental frequency of $2\omega_o$, and therefore double the generator frequency. Consequently, $k_1 = k_3 = k_5 = 0$ in Equations (4) and (5). Furthermore, the coefficients of Equations (4) and (5) produce the following:

$$\left.\begin{aligned} L_o &= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\,\Phi_e}{d\,i_1} \bullet d\,(\omega_o\, t) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{L_m\, d\,(\omega_o\, t)}{1 + q^2\, J_1^2 \bullet \sin^2 \omega_o\, t} + L' = L' + \frac{L_m}{\sqrt{1 + q^2\, J_1^2}} \end{aligned}\right\} \tag{8}$$

$$\left. k_n = \frac{1}{\pi} \int_0^{2\pi} \frac{d\,\Phi_e}{d\,i_1} \bullet \cos n\,\omega_o\, t \;\; d\,(\omega_o\, t) \quad \ldots\; n = 2, 4, 6 \ldots \right\} \tag{9}$$

From Equation (9) there follows

$$\left.\begin{aligned} k_2 &= \frac{4\, L_m}{q^2\, J_1^2} \bullet \left\{ \frac{1 + \frac{q^2 \bullet J_1^2}{2}}{\sqrt{1 + q^2\, J_1^2}} - 1 \right\} \\ k_1 &= \left(1 + \frac{2}{q^2\, J_1^2}\right) k_2 - \left(L_o - L'\right) \\ k_n &= \left(1 + \frac{q^2\, J_1^2}{2}\right) = \frac{q^2\, J_1^2}{4} \left(k_{n-2} + k_{n+2}\right) \end{aligned}\right\} \tag{10}$$

The sin terms do not emerge in the Fourier series (since $i_g = 0$). Therefore, we obtain the result:

$$\frac{d\Phi_e}{d\,i_1} = L_o + k_2 \cos 2\,\omega_o t + k_4 \cos 4\,\omega_o t + \ldots \tag{11}$$

and consequently

$$\frac{d}{dt}\left(\frac{d\Phi_e}{d i_1}\right) = -\omega_o \left\{2k_2 \sin 2\omega_o t + 4k_4 \sin 4\omega_o t + ...\right\} \quad (12)$$

The values of $\frac{L_o}{L_m}, \frac{k_2}{L_m}, \frac{k_4}{L_m}$ are plotted in Figure 2 as functions of q and J_1.

The value of the inductance L_e of the iron-cored coil, defined according to Schunk-Zenneck,[iii] is obtained from L_o and k_2: $\omega_o L_e J_1 \cos \omega_o t$ is the fundamental oscillation of the coil voltage if the current in the coil is $i_1 = J_1 \sin \omega_o t$. Since the total voltage across the coil is $\frac{d\Phi_e}{d i_1} \bullet \frac{d i}{d t}$ it follows that

$$\omega_o L_e \bullet J_1 = \frac{1}{\pi} \int_0^{2\pi} \frac{d\Phi_e}{d i_1} \bullet \frac{d i_1}{d t} \cos \omega_o t \, d(\omega_o t) = $$
$$= \frac{\omega_o J_1}{\pi} \int_0^{2\pi} \frac{d\Phi_e}{d i_1} \left(\frac{1 + \cos 2\omega_o t}{2}\right) d(\omega_o t) \quad (13)$$

By comparing expressions (8) and (9) with Equation (13) one obtains:

$$L_e = L_o + \frac{k_2}{2} \qquad \text{(see Figure 2)} \qquad (14)$$

$(L' + L_m)$ is the maximum value of L_e and of L_o.

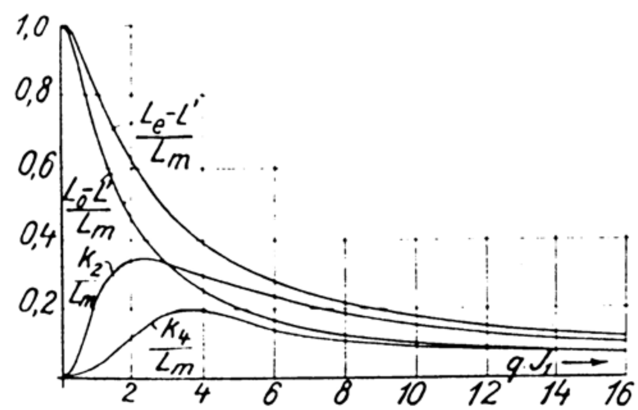

Figure 2. The first coefficients of the Fourier Series for $d\Phi_e/di_1$ in an iron-core, without DC bias-magnetization, as a function of the amplitude of the magnetizing sinusoidal AC.

b) L_o and k_n in the case of DC bias-magnetization ($i_g \neq 0$).

For the case of DC bias-magnetization, it must be confessed that the dependence of field strength and flux in iron is substantially more complicated, since they are rendered through the formulation of Equation (6). Here, an analytical calculation of the coefficients becomes specified by Equation (4). Nevertheless, qualitative assertions can be made without the assumption of a specific magnetization curve:

1. In the presence of bias-magnetization, the fundamental frequency of the function $d\Phi_e/di_1$ is equal to the generator frequency ω_o ; then here, in contrast to being asymmetric

$$\Phi_e(i_g + i_1) \neq -\Phi_e(i_g - i_1)$$

the quantity $d\Phi_e/di_1$ can therefore be developed in the series

$$\frac{d\Phi_e}{d i_1} = L_o + k_1 \sin \omega_o t + k_2 \cos 2\omega_o t + k_3 \sin 3\omega_o t + \ldots \quad . \tag{15}$$

[The terms with cos $(2n - 1)\omega_o t$ and sin $2n\omega_o t$ $(n = 1, 2, \ldots)$ are neglected; then, in the case of the superposition of DC and sinusoidal AC ($i_1 = J_1 \sin \omega_o t$), the voltage of an iron coil can be represented by the following series[6] (neglecting hysteressis and AC breakdown):

$$\frac{d\Phi_e}{d i_1} \bullet \frac{d i_1}{d t} = p_1 \cos \omega_o t + p_2 \sin 2\omega_o t + p_3 \cos 3\omega_o t + \ldots$$

which follows immediately from the ingredients of Equation (15).]

2. Here, for the first time, the interesting new coefficients k_1, k_3, … emerge from Equation (15), as is easily seen. They are zero if $i_g = 0$ or $J_1 = 0$ (also see), or if $i_g = \infty$ or $J_1 = \infty$. For that reason, it is to be expected that the values of k_1, k_3, … pass through maxima for the case of constant i_g and variable J_1 as well as for variable i_g and constant J_1.

II. Spontaneous Self-Excitation of Oscillations at the Generator Frequency and Their Integral Multiples.

(Iron-core coils with pure AC magnetization.)

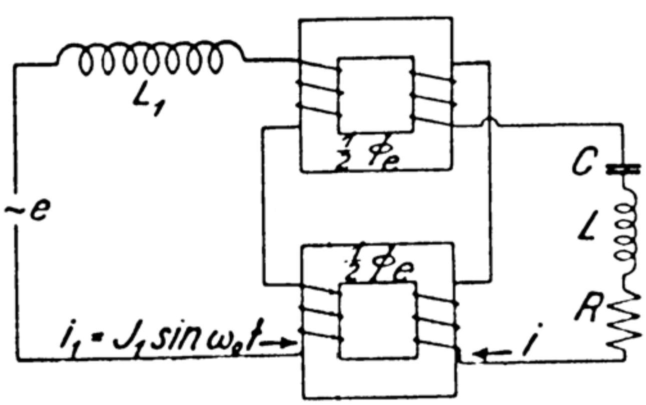

Figure 3. Double core connection without superposed DC.

6 See: R. Strigel, this Jahrbuch, Vol. 29, p. 10, 1927.

Clearly, as one contemplates the circuit diagram shown in Figure 1, one recognizes in the double-core circuit (Figure 3), already employed by K. Heegner,[7] the essence of the self-excitation phenomena: 2 equivalent closed iron transformers (let the magnetic flux of each transformer be $\frac{1}{2}\Phi_e$) have series-wired primaries connected to a generator at frequency ω_o through an arbitrary, very large choke coil, L_1. Its secondary windings are interconnected in opposition through an oscillation circuit (L, C, R). By this, a periodic change in the inductance in the secondary circuit is, accordingly, achieved at the frequency of the primary current, $i_1 = J_1 \sin \omega_o t$, while the voltage induced by the primary current is counteracted.

From section I, Equations (3) and (11), if the series for $d\Phi_e/di_1$ is truncated after the second term, the differential equation for the current in the secondary circuit is

$$(L_o + k_2 \cos 2\omega_o t)\ \frac{di}{dt} + L\frac{di}{dt} + i\,(R - 2\,\omega_o k_2 \sin 2\omega_o t) + \frac{1}{C_1}\int i\,dt = 0 \quad . \tag{1}$$

According to Rayleigh, this equation possesses the particular integral

$$i = A \bullet \sin\left(\omega_o t - \phi\right) \tag{2}$$

assuming that L is sufficiently large that the harmonics of i can be neglected, and, furthermore, that the following tuning conditions of the secondary are satisfied.

$$\left.\begin{aligned} \left(\omega_o L - \frac{1}{\omega_o C}\right) &= -\omega_o\left(L_o - \frac{k_2}{2}\bullet\cos 2\phi\right) \quad & a \\ R &= -\,\omega_o \bullet \frac{k_2}{2}\bullet \sin 2\phi \quad & b \end{aligned}\right\} \tag{3}$$

(The inductance of the primary is chosen to be so large that the reaction in the primary to the oscillations in L, C, R can be neglected.)

Equations (3a) and (3b) result from inserting Equation (2) into Equation (1) and asserting the following: self-excited oscillations commence, in the secondary of the circuit connection shown in Figure 3, at the generator frequency, if (a) $\omega_o L = \frac{1}{\omega_o C}$ (assuming $L_o \ll L$), and if (b) the energy condition, Equation (3b), is satisfied. Accordingly, the oscillations are only possible if $R \le \omega_o \frac{k_2}{2}$ ($\sin 2\varphi < 0$). From the relationship of k_2 to the amplitude of the magnetization current, J_1, depicted in Figure 2, it is to be concluded that oscillations in the secondary, below as well as above certain values of J_1, cannot be maintained.

If the development of section I Equation (11) is truncated after the first three terms, then the differential equation for the secondary current reads:

[7] Zeitschrift für Physik, Vol. 33, p. 85 and following, 1925.

$$\begin{aligned} &(L_o + k_2 \cos 2\omega_o t + k_4 \cos 4\,\omega_o t + L\,)\ \frac{d\,i}{d\,t} \\ &+\ (R - 2\,\omega_o k_2\ \sin 2\omega_o t - 4\,\omega_o k_4\ \sin 4\omega_o t)\ i\ +\ \frac{1}{C_1}\int i\,dt \ =\ 0 \end{aligned} \tag{4}$$

with the particular integral[8]

$$i \ = \ A\bullet\sin\left(2\,\omega_o\,t - \phi\right) \tag{5}$$

provided these conditions are met:

$$\left.\begin{aligned} \left(2\,\omega_o\,L - \frac{1}{2\,\omega_o\,C}\right) &= -2\ \omega_o\left(L_o - \frac{k_4}{2}\bullet\cos 2\phi\right) \\ R \ &= \ -\ 2\,\omega_o\bullet\frac{k_4}{2}\bullet\sin 2\phi \end{aligned}\right\}\begin{matrix} a \\ \\ b \end{matrix}\quad . \tag{6}$$

According to this, an octave (doubling) of generator oscillation is excited when the secondary circuit is tuned approximately to the frequency $2\omega_o$. Since, from Figure 2, k_4 , like k_2 , is a function of the amplitude of the magnetization current J_1, it follows that Equation (6) can, again, be fulfilled only within a specific range of J_1.

Finally, if the entire series from section I Equation (11) is inserted into the differential equation for the secondary current, then, in the same manner, there result the conditions for the establishment of an oscillation in the secondary circuit whose frequency is any even or odd integral multiple of the generator frequency.

III. Spontaneous Self-Excitation of Oscillations at Half the Generator Frequency and Their Integral Multiples.

(Iron-core coils with AC and DC magnetization.)

The circuit configuration shown in Figure 4 differs from the similar one in Figure 3 only in that a third winding, in which a DC current flows, is arranged on the iron core. According to Ib the series

$$\frac{d\Phi_e}{d\,i_1} \ = \ L_o \ + \ k_1 \sin\omega_o t \ + \ k_2 \cos 2\,\omega_o\,t \ + \ k_3 \sin 3\,\omega_o\,t \ + \ldots \quad . \tag{1}$$

is valid for the time-dependence of the function $d\Phi_e/di_1$. The variation of the inductance of the secondary circuit has the same period as the current of the generator circuit, $i_1 = J_1 \sin \omega_o t$.

The differential equation for the current in the secondary circuit is, therefore,

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[8] As usual, in the following it is assumed that L is sufficiently large, so that the Fourier series for i produces a term that predominates against all others.

$$(L_o + k_1 \sin \omega_o t + ... + L) \, \frac{d\,i}{d\,t} \; + \; (R + \; \omega_o k_1 \cos \omega_o t - 2\,\omega_o k_2 \, \sin 2\omega_o t \, + ...) \, i$$
$$+ \; \frac{1}{C_1} \int i \, dt \;\; = \;\; 0 \quad . \qquad (2)$$

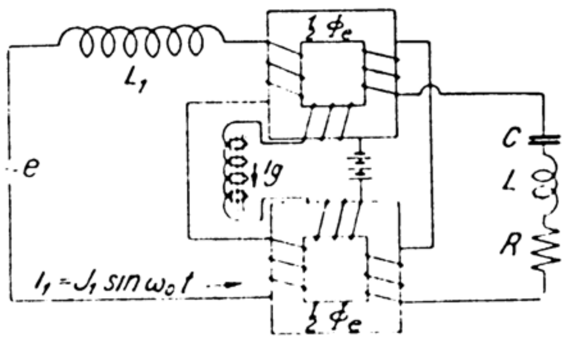

Figure 4. Double core connection with DC superposition.

Under the same presuppositions as in the previous section it follows from this equation that an oscillation is established in the secondary circuit at half the generator frequency ($\omega_o/2$), or an integral multiple of it ($n\omega_o/2$), if the following stipulation equations are satisfied:

$$\left. \begin{aligned} \left(n \frac{\omega_o}{2} L - \frac{1}{n \frac{\omega_o}{2} C} \right) &= - n \frac{\omega_o}{2} \left(L_o - \frac{k_n}{2} \bullet \cos 2\,\phi \right) \quad (a) \\ R \;\; &= \;\; - \; n \frac{\omega_o}{2} \bullet \frac{k_n}{2} \bullet \sin 2\,\phi \quad (b) \end{aligned} \right\} \qquad (3)$$

that is, when the L, C, R circuit is tuned approximately to the frequency $n\omega_o/2$ and the dissipation resistance satisfies the requirement:

$$R \;\; \leq \;\; n \frac{\omega_o}{2} \bullet \frac{k_n}{2} \bullet \left(k_n - f(J_1 \bullet i_g) \right) \quad .$$

(For n = 1 [frequency halving] the analogy with the mechanical problem mentioned in the introduction is complete.)

Experiments for Sections II and III.

The Oscillogram shown in Figure 5[iv] is an example of the self-excited oscillations treated in Section II: the well-known frequency doubling of iron cores having no bias-magnetization. The circuit connection was that of Figure 3; the generator frequency was $\omega_o = 2\pi\cdot 1000\ sec^{-1}$; the secondary circuit (without an iron core) was tuned a little higher than $2\omega_o$.

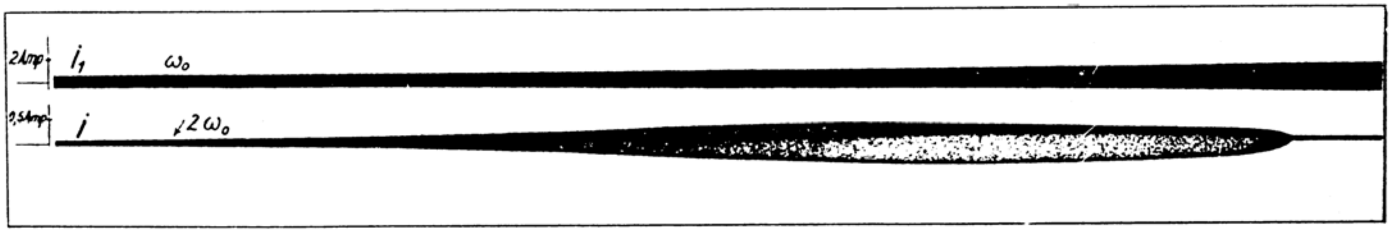

Figure 5. Frequency doubling with the double core connection. The amplitude of the magnetization current, i_1, gradually increased during the recording.

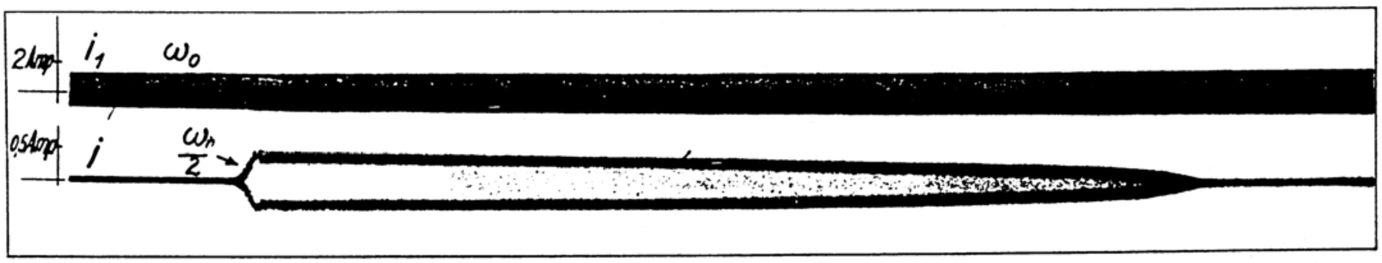

Figure 6. Likewise for frequency-halving.

The generator voltage, and therefore the amplitude J_1 of the magnetizing current i_1, gradually increased during the recording. Certain values of the amplitude (and therewith of k_2, compare Figure 2) established oscillations i in the initially dead circuit L, C, R at the frequency $2\omega_o = 2\pi\cdot 2000\ sec^{-1}$, the latter vanished again at yet higher values of J_1 (in harmony with the attached discussion of Equation (6) in section II).

As an example of section III, the oscillogram of Figure 6 shows the familiar frequency-halving for iron cores with bias-magnetization. The circuit is shown in Figure 4. The three windings have an equal number of turns ($i_g = 2$ amps). The L, C, R circuit was approximately tuned to the frequency of $\frac{1}{2}\ \omega_o$ ($\omega_o = 2\pi\cdot 1000\ sec^{-1}$). As in the previous example, oscillations i appear in the secondary circuit (this time at half the generator frequency) only within a certain region of the amplitude J_1 of the magnetization current i_1. This is also in agreement with the theory (compare Equation (3) of Section II) since, according to the considerations of section I for a constant DC current i_g, k_1 must possess a similar course as k_2 in Figure 2.

According to all these experiments, the tuning of the generator circuit is unimportant as long as $i \ll i_1$. In our case, a very large self-inductance is connected between the generator and the primary of the transformers.

IV. Self-Excitation of Independent Oscillations in Coupled Systems.

In addition to the self-excited phenomena that were treated in the previous sections for the case of two coupled oscillation circuits, one of which contains an iron-cored coil, two oscillations with the frequencies ω_1 and ω_2 ($\omega_1 \neq \omega_2 \neq \omega_o$) emerge, by which the resulting current amplitudes are developed under pendulum conditions.[9,v]

Here, an extension of Rayleigh's formulation, derived from the periodicity of the self-induction, also makes possible the establishment of these superposed oscillations (in fact the oscillations for which, according to Heegner, the frequency relation $n\omega_o = \omega_1 + \omega_2$, for n = 1, 2, …, exists). The important practical case, $2\omega_o = \omega_1 + \omega_2$ (an iron-core with pure AC magnetization), will be treated in the following example (see Figure 7).

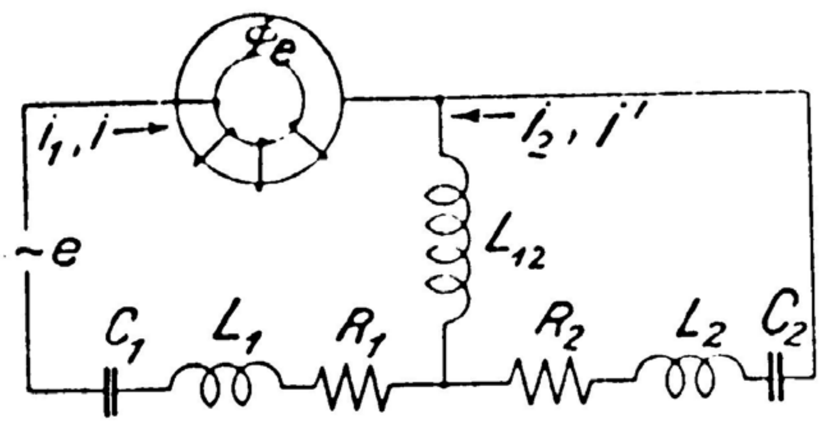

Figure 7. Coupled circuit which, under certain conditions, can excite oscillations given by the frequency relation $2\omega_o = \omega_1 + \omega_2$.

For this circuit connection the requirement again exists that the iron-free inductances are supposed to be large in comparison to the iron-cored coil.

Let the generator voltage be: $e = E \sin(\omega_o t - \psi)$.

The equations of the system:

$$\left.\begin{aligned} &\frac{d\Phi_e}{di_1} \bullet \frac{di_1}{dt} + L_1 \bullet \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1\, dt + L_{12}\frac{d(i_1+i_2)}{dt} = e \quad (a) \\ &L_2 \bullet \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2\, dt + L_{12}\frac{d(i_1+i_2)}{dt} = 0 \quad (b) \end{aligned}\right\} \quad (1)$$

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[9] In addition to the work of Heegner, see: Plendl, Sammer, Zenneck, this journal, Vol. 26, p. 104, 1925.

possess, first of all, the solution $i_1 = J_1 \sin \omega_o t$, which, again, is thus obtained for the case that the iron-core coil is replaced by the Schunck-Zenneck defined inductance, L_e. As with the uncoupled system, we arrive from Equation (1) at the equations of the self-excited oscillations of this system, in that the small oscillations of the primary current i ($i \ll i_g$) are superposed on the secondary current i'. The result is:

$$\left.\begin{aligned} &\frac{d\Phi_e}{d i_1}\bullet\frac{d i_1}{dt} + i\bullet\frac{d}{dt}\left(\frac{d\Phi_e}{d i_1}\right) + L_1\bullet\frac{di}{dt} + R_1 i_1 \\ &\qquad + \frac{1}{C_1}\int i\,dt + L_{12}\frac{d(i+i')}{dt} = 0 \\ &L_2\bullet\frac{di'}{dt} + R_2 i' + \frac{1}{C_2}\int i'\,dt + L_{12}\frac{d(i+i')}{dt} = 0 \end{aligned}\right\} \begin{matrix} a \\ \\ b \end{matrix} \qquad (2)$$

If the expression for $\frac{d\Phi_e}{d i_1}$ in section I Equation (11) is truncated after the second term, and then is placed into the previous equation, one obtains

$$\begin{aligned} \frac{d\Phi_e}{d i_1} &= L_o + k_2 \cos 2\omega_o t \qquad \text{and} \\ \frac{d}{dt}\left(\frac{d\Phi_e}{d i_1}\right) &= -2\omega_o k_2 \sin 2\omega_o t \end{aligned} \qquad (3)$$

Equation (2) will be satisfied by the expression:

$$i = A\bullet\left\{\sin(\omega_1 t - \phi_1) + \sin(\omega_2 t - \phi_2)\right\} \qquad (4)$$

(therefore, by two stationary oscillations at different frequencies) under the following conditions:[10]

[10] Small damping is presupposed in b2 and d2. - X_ω is the reactance of the system referred to the primary circuit, without an iron coil. R_ω is the loss resistance referred to the primary circuit of the like system (for frequency ω).

$$\left.\begin{aligned}
a)\quad & \omega_o = \frac{\omega_1+\omega_2}{2}\\
b)\quad & X_{\omega_1} = -\omega_1\left(L_o - \quad a\bullet\frac{k_2}{2}\cos(\phi_1+\psi_2)\right)\\
& X_{\omega} = \left(\omega L_1 - \frac{1}{\omega C_1}\right) + \frac{\omega L_{12}\bullet\left(\omega L_2-\frac{1}{\omega C_2}\right)}{\left(\omega(L_{12}+L_2)-\frac{1}{\omega C_2}\right)}\\
c)\quad & X_{\omega_2} = -\omega_2\left(L_o - \frac{1}{a}\frac{k_2}{2}\sin\,(\phi_1+\phi_2)\right)\\
d)\quad & R_{\omega_1} = -\omega_1\bullet a\bullet\frac{k_2}{2}\sin\,(\phi_1+\phi_2)\\
& R_{\omega} = R_1 + \frac{\omega^2 L_{12}^2\, R_2}{\left(\omega(L_{12}+L_2)-\frac{1}{\omega C_2}\right)^2}\\
e)\quad & R_{\omega_2} = -\omega_2\bullet\frac{1}{a}\bullet\frac{k_2}{2}\sin\,(\phi_1+\phi_2)
\end{aligned}\right\}\qquad (5)$$

Equations (5, a-e) immediately result by the insertion of Equation (4) into Equation (2).

The flow of secondary current follows immediately from Equation (2b) if the expression given by Equation (4) for i is inserted into it.

Discussion of Equation (5):

1. The X_n quantities on the left sides of Equations (5b) and (5c) represent the reactance of the system without an iron coil referred to the primary circuit for frequencies ω_1 and ω_2, respectively. By assumption, when the iron-cored coil is small in comparison to the iron–free coil it should follow that ω_1 and ω_2 are approximately the natural or eigenfrequencies of the system without the iron coil; therefore the generator must possess one frequency, which lies in the middle between these eigenfrequencies.
2. From (1) it follows that the frequency of the beats, which result from these self-excited oscillations (swings of current amplitudes), must be brought into harmony with experience, the tighter the coupling between both circuits the higher the beat frequency must be.
3. The energy balance of the system is contained in Equations (5.d, e). First of all, they result in amplitude-ratios of both self-excited oscillations: $a^2 = \frac{R_{\omega_1}}{R_{\omega_2}}\bullet\frac{\omega_2}{\omega_1}$ (in agreement with Heegner). Furthermore, it follows from this that (analogous to uncoupled systems) the oscillations ω_1 and ω_2 cannot be maintained below certain values of the parameter k_2 , $\left(k_2 < \frac{1}{a}\bullet\frac{2}{\omega_2}\bullet R_{\omega_2}\right)$, that is, with regard to the dependence rendered in Figure 2, $k_2 = f(J_1)$, (in accordance with experiments),[11] the oscillations of ω_1 and ω_2 vanish above and below certain vales of the magnetization current, J_1.

[11] Plendl, Sammer, Zenneck, this Jahrbuch, Vol. 26, p. 104, 1925.

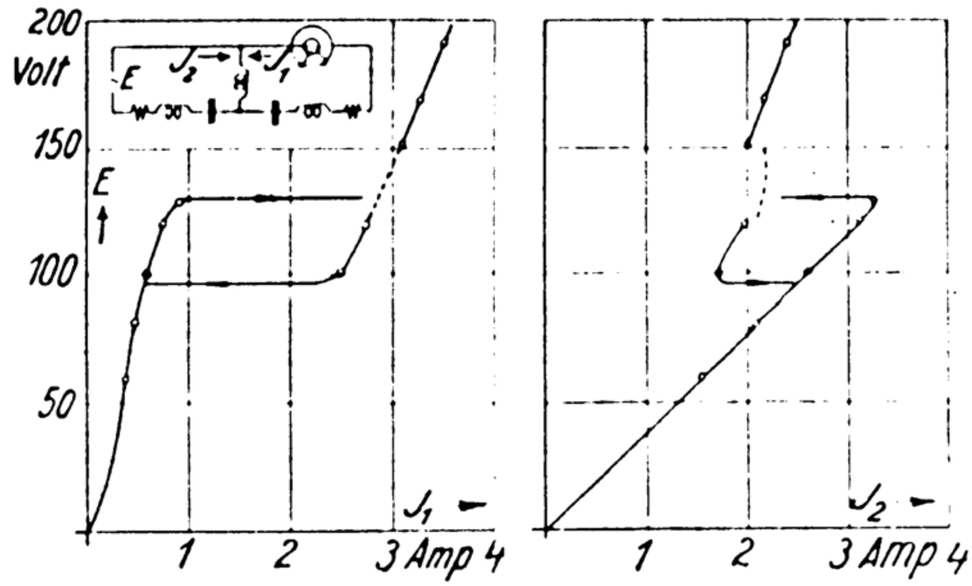

Figure 8. Example of a voltage-current characteristic for the coupled-circuit connection shown in Fig. 7. The region of self-excitation is dotted.

4. Finally, it follows from the previously developed Equations (1) and (2) that, for the establishment of self-excited oscillations, it is insignificant whether the generator lies in the primary circuit or the secondary circuit of Figure 7, assuming that the generator voltage is adjusted so that the amplitude of the magnetization, J_1, is the same value in both cases.

This latter point should be illustrated with a few brief experiments:

The curves in Figure 8 are taken from a similar circuit that Plendl, Sammer, and Zenneck[12] have used. The iron coil here was placed in the secondary and, because of that, possesses the characteristic shape in which the generator current jumps backward when increasing the generator voltage from smaller values. At certain generator frequencies these jump phenomena are superposed on the above-mentioned beats.[13] (The dotted parts of the curves.) (Plendl, Sammer, and Zenneck trace these well-known swings, which are connected with the jumps of the magnetization current J_1, back to the sudden variation of the reaction of the secondary on the primary.)

[12] Loc cit.
[13] Analogous to the beats shown in oscillogram (10).

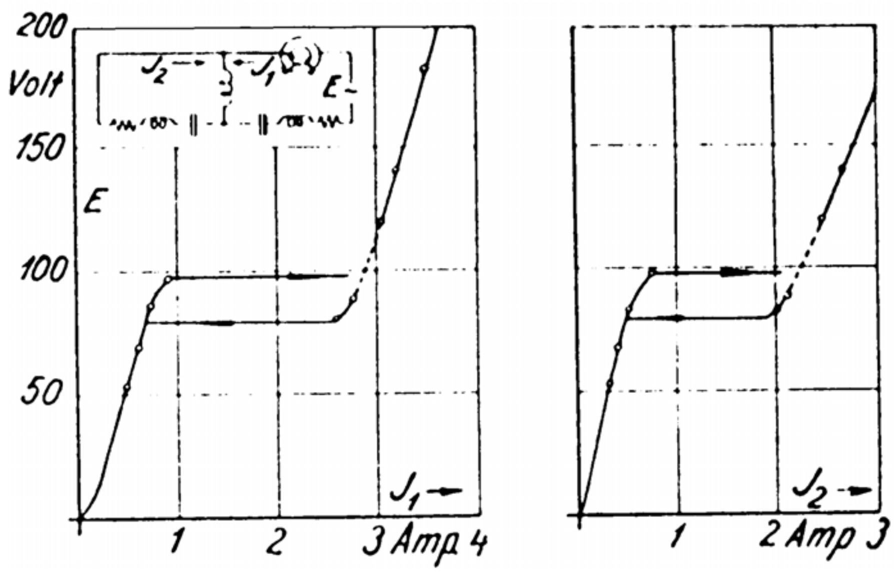

Figure 9. The circuit and generator frequencies are the same as in Fig. 8, nevertheless, the generator lies in another part of the hookup.

The trajectories taken in the curves of Figure 9 are completely identical to the trajectories of Figure 8. The circuit differs from the previous one only in that the generator is directly connected to the windings of the iron coil. The reaction of the present secondary circuit on the generator is naturally independent of the current amplitude and can usually be described by a constant auxiliary impedance in the generator circuit. Therefore, it is to be expected that the current-voltage characteristic of the generator (see Figure 9) does not differ from that of a simple oscillation circuit with an iron coil. As our theory demands, it appears that the equal beats (dotted parts of the curves) are actually only the case for identical values of magnetization current J_1 and identical generator frequencies in the circuits of Figures 8 and 9. (An explanation for these swings, which return on the characteristic of the effective values, seems not to be possible for the latter circuits, since these kinds of swings with iron cores do not occur in simple harmonic motion.[vi])

Finally, oscillogram (10) shows that the jump phenomena can occur completely independent of pendulums. The circuit was the same as in Figure 8 except the coupling was a little tighter. During the photographic exposure, the generator-voltage was brought up from smaller to higher values: next the normal jump of current appeared, and after further increasing the voltage to higher values (and thereby J_1) beats appeared, which again extinguished for yet greater values of magnetization current.

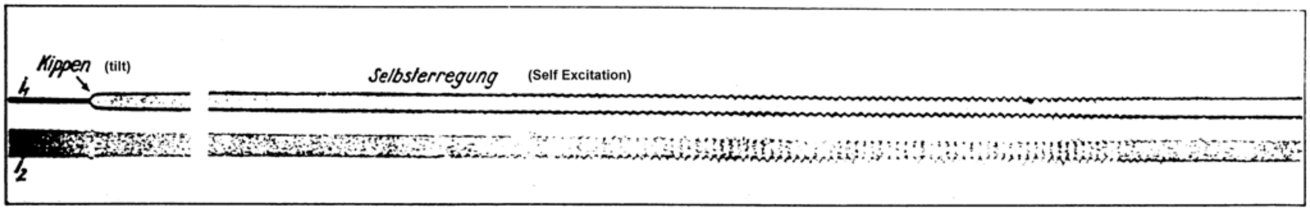

Figure 10. Self-excitation in the circuit of Fig. 7. During the photograph the amplitude of the generator voltage gradually increased. [Kippen = jump; Selbsterregung = self-excited]

Conclusions.

The foregoing work has demonstrated the fact that, in first approximation, self-excited oscillations in circuits with iron-cored coils are the same as those that occur in circuits that possess coils whose inductance is varied periodically by a mechanical device instead of possessing iron coils. (In place of $\frac{d\,\Phi_e}{d\,i_1}$ in Section I, Equation (3) there is an L(t).)

The foregoing theory makes no assumption about the amplitudes of the self-excited oscillations. (In Equation (2) of section II, the quantities A, etc., are arbitrary constants of integration.) The basis for this rests on the fact that, as Rayleigh has already emphasized, the effect of the resulting oscillations on the coefficients in differential equation (3) of section I was not considered. (It was assumed that $i \ll i_1$.) Furthermore, on the same basis, the theory of non-independent self-excitation is impractical in the form communicated by K. Heegner.

Let me express my deepest gratitude to Herr Councilor Zenneck for the stimulating interest that he has taken in this work.

Summary.

In the foregoing work an analytical presentation of K. Heegner's generally coherent description of stationary self-excitation in oscillation circuits with iron-cored coils was traced back to a well-known form from mechanics:

With the help of the method of small oscillations it was shown, above all, that, for the case of sinusoidal magnetization currents, the iron coils in an oscillation circuit whose self-excitations were investigated, can be replaced by a periodically varying inductance. The fundamental frequency of this inductance variation is, consequently,

1. Equal to double the frequency of the generator (in the case of pure AC magnetization).
2. Equal the generator frequency (in the case of DC plus AC magnetization).

The amplitudes of the fundamental and harmonic oscillations of these periodic inductance variations are functions the effective values of the magnetization current.

For uncoupled systems, this formulation leads to a well-known differential equation, whose solution (for a certain choice of coefficients), according to Rayleigh, possesses stationary oscillations at a frequency equal to half the frequency of the inductance variation or to an integer multiple of it. That is to say, in Case 1 the self-excitations superposed on the magnetizing current possess a frequency equal to the generator frequency, and in case 2 a frequency equal to half the generator frequency or an integer multiple of it.

The equations of constraint for the establishment of self-excitation are easy to derive and result in quantitative statements concerning the necessary tuning of the system and amplitudes of the magnetizing currents.

In closing, an example was shown that, in a similar manner, could derive the requirements which exist for the establishment of self-excitations to appear in the case of coupled systems. (Swings of current amplitudes.)

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(Submitted: 20 April 1929.)

Comments and References Added by Translator

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i As is pointed out by many authors (see Cunningham, Nonlinear Analysis, 1958, p. 278, for example) Rayleigh's expression may be recast into Mathieu's equation by the following procedure:

1. Rayleigh's expression: $\frac{d^2 u}{d t^2} + \alpha \frac{du}{dt} + n^2 \bullet u - 2\beta \sin \omega_o t \bullet u = 0$
2. To remove the term in the first derivative, make the change of variable: $u = y\, e^{-\alpha t/2}$.
3. The equation now becomes: $\frac{d^2 y}{d t^2} + \left(n^2 - \left(\frac{\alpha}{2}\right)^2 - 2\beta n^2 \sin \omega_o t \right) y = 0$
4. Then take

$$\omega_o t = 2z - \frac{\pi}{2}$$

$$\frac{dy}{dt} = \frac{\omega_o}{2}\left(\frac{dy}{dz}\right)$$

$$\frac{d^2 y}{dt^2} = \left(\frac{\omega_o}{2}\right)^2 \left(\frac{d^2 y}{dz^2}\right)$$

5. By making these changes, the differential equation becomes

$$\frac{d^2 y}{d z^2} + \frac{4}{\omega_o^2}\left(n^2 - \left(\frac{\alpha}{2}\right)^2 - 2\beta n^2 \cos 2z \right) y = 0$$

6. Then make the following definitions

$$a = \frac{4}{\omega_o^2}\left(n^2 - \left(\frac{\alpha}{2}\right)^2 \right)$$

$$q = \frac{2\,(2\beta)\, n^2}{\omega_o^2}$$

7. The resulting representation has now recast Rayleigh's expression into the form of Mathieu's Equation:

$$\boxed{\frac{d^2 y}{d z^2} + \left(a - 2q \cos 2z\right) y = 0} \quad .$$

ii Dreyfuss, L., Electrotechnik und Maschinenbau, 1911. (See: N. Minorsky, Nonlinear Oscillations, Van Nostrand, 1962, p. 64; A.A. Andronov, A.A. Vitt and S.E. Khaikin, Theory of Oscillators, Dover, 1987, pp. 119-112.) Also see Dreyfuss, L., Arch. fur Electrotechn., Vol. 2, 1913, p. 343.

iii Schunk, H., and J. Zenneck, "Uber Schwingungskreise mit Eisenkernspulen, (On Oscillation Circuits with Iron Coils)" Jahrbuch der drahtlosen Telegraphie (Zeitschrift für Hochfrequenztechnik),Vol. 19, 1920, p. 170.

iv The oscillogram figures are high-resolution scans of first generation Xerox copies of the journal microfilm, and witness to the tragic consequence of latter 20th century "library science"!

v For the transient case for *linear* coupled coils, compare the analyses of:

a. Page, L., and N.I. Adams, Principles of Electricity, van Nostrand, 1931, pp. 504-511. (Page and Adams also treat the driven case, pp. 519-525.)
b. Smythe, W.R., Static and Dynamic Electricity, McGraw-Hill, 2nd edition, 1950, pp. 340-346. (Note the error in Equation (14) on p. 345.)
c. Skilling, H.H., Transients in Electric Circuits, McGraw-Hill, 2nd edition, 1952, pp. 220-230.

vi The text reads "Schw.-Kr.", which is a German abbreviation, used only once in the text, and for which no translation was found. Perhaps it is "Schwingungen Kraften" (oscillating forces). We assume it is probably equivalent to the english "SHM" (Simple Harmonic Motion).